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ARITHMETICAL COMBINATIONS

J. H. SMITH

Superintendent of Schools, Belvidere, Illinois

THE RELATIVE DIFFICULTY OF THE ADDITION COMBINATIONS

One of the large problems confronting us in the scientific development of education is the preparation of material for corrective work and the development of new methods of instruction, based upon experimentation. The following study was made with a view to carrying out this program in one field of elementary instruction. It was the writer's opinion that many pupils make low scores in arithmetic tests because they are deficient in a few difficult combinations rather than equally deficient in all combinations. If we can locate the difficult combinations and concentrate drill on them rather than spend time on the combinations already fairly well mastered, we shall be able to make more rapid progress. The following investigation was made for the purpose of determining the difficult combinations in addition.

The apparatus consisted of a frame similar to a tachistoscope in which a dropping screen exposed a card on which one of the combinations had been printed. At the instant the screen was tripped by the latch the experimenter started a stop watch. The subject gave the sum as quickly as possible. As soon as the subject began to vocalize the sum, the watch was stopped. The combination was then withdrawn, and the time recorded on the back of the card. The screen was raised, another combination was inserted, and the experiment was continued as before.

It will be noted that the time required for the subject to give the sum of any combination involves several factors: (1) the time required for the screen to drop, (2) the time of recognition of the two digits on the card, (3) the time of recall of the association of these digits with their sum, and (4) the reaction time for the vocalization of the sum.

After each subject had given the responses to the eighty-one arrangements of the forty-five combinations, he was required to

name a single figure exposed on a card in the same way. Nine cards, containing the figures from 1 to 9, were exposed in a miscellaneous order. The time of one of these responses involves the dropping of the screen, the recognition of a digit, and the reaction time of vocalization. No appreciable variation was found in the time of response of any individual to the different cards and very slight variations between the responses of different individuals. It is clear, then, that the *time required for the recall of the association of the sum and the digits* is the variable factor in the time of responses to the various combinations. Those combinations for which this association is most easily recalled will have the shortest time of response, and the difficult ones will require the longest time.

Eighty-six pupils from the third, fourth, fifth, and seventh grades of the training school and one arithmetic-methods class of the State Normal School at Whitewater, Wisconsin, were tested in this manner.

The median time responses for the various combinations were then computed for each of the five groups. The combinations which proved difficult for the third- and fourth-grade groups also proved difficult for the upper-grade groups and for the adult class. In order to eliminate any minor variations, the final median time responses were computed from the medians of the five groups.

In carrying out the experiment care was taken to determine the effect of the arrangement of any two digits on the time response. For example, will it take longer to recall the sum, 14, for the arrangement $\frac{5}{9}$ than for the arrangement $\frac{9}{5}$? It was found that in nineteen of the combinations the time of response was slightly greater when the larger digit was above; in fourteen combinations the time of response was slightly greater when the smaller digit was above, and in three cases the time was the same for the two arrangements. These facts indicate that the arrangement of the digits in a combination does not seem to influence greatly the time required for giving the sum. The average of the medians for the two arrangements of each combination was taken as the time response for that combination.

The various combinations were then arranged in order from the one with the shortest time of response to the one requiring the

longest time. The forty-five combinations were then divided into three equal groups on this basis, which for the purposes of this discussion will be designated the easy group, the medium group, and the difficult group. Table I shows the three groups

TABLE I

EASY GROUP		MEDIUM GROUP		DIFFICULT GROUP	
Combination	Response Time in Seconds	Combination	Response Time in Seconds	Combination	Response Time in Seconds
5+5	1.07	6+4	1.23	6+3	1.42
4+1	1.13	3+3	1.23	6+7	1.42
5+1	1.13	8+8	1.23	8+7	1.43
6+1	1.13	7+7	1.25	7+4	1.44
1+1	1.14	6+6	1.26	9+2	1.46
4+4	1.14	9+9	1.27	8+5	1.47
2+1	1.14	6+2	1.27	9+4	1.50
8+1	1.15	7+3	1.27	7+5	1.50
9+1	1.16	8+2	1.28	9+8	1.51
2+2	1.17	4+2	1.28	8+3	1.51
3+1	1.18	5+2	1.31	9+5	1.53
7+1	1.19	6+5	1.34	8+6	1.54
4+3	1.19	5+3	1.37	9+6	1.57
5+4	1.22	8+4	1.39	9+3	1.57
3+2	1.22	7+2	1.40	9+7	1.81

and the time of response for the various combinations. It appears from this table that all the easy combinations have sums that do not exceed 10; that with one exception the sums of the very difficult combinations exceed 10; and that the doubles are relatively easy since all of them fall in the easy group or the first half of the medium group.

Some of the pupils gave interesting reactions when questioned about the process that they used in obtaining the sums of combinations requiring long time responses. One girl, who required an especially long time to give the sum of 5 and 7, explained the process in this way: "5 and 5 are 10 and 2 are 12." She was a specialist on doubles and split the 7 into a 5 and a 2 so that she might have a 5 and 5 combination to which she could then add the remaining 2. This required a subtraction and two additions, thus explaining the long time required for her response to that combination. She added 6 and 9 as follows: "6 and 6 are 12 and 3 are 15." She gave similar reactions to 5 and 8; 7 and 9; etc. Another pupil required a long time for the sum of 6 and 9. He

explained his process as follows: "6 and 10 are 16; 6 and 9 are 1 less than 6 and 10; then 6 and 9 are 15." He had to think through a similar form every time any number was added to 9 and of course gave much slower responses. Another pupil could not give the sums directly but had to go through the form of saying "5 and 9 are 14" instead of merely saying "14." These individual reactions are significant from the standpoint of methods of teaching. We should be careful about letting pupils acquire forms or roundabout schemes for securing a result in the lower grades which will prove a handicap to them in the upper grades.

The writer also tested an expert in rapid calculation and found practically no variation in the time of his responses to different combinations. This represents the ideal toward which we should strive in teaching the combinations—to master equally well all of the combinations. The forty-five combinations were arranged into three drills so that this equal mastery could be achieved. The two arrangements of each combination of two different digits are given, and each double is repeated so that equal stress will be placed upon all combinations in the same group.

DRILL A—EASY COMBINATIONS

4	9	3	8	1	4	1	1	3	4	1	6	1	2	5
5	<u>1</u>	4	<u>1</u>	3	4	4	<u>8</u>	<u>2</u>	3	5	<u>1</u>	<u>1</u>	<u>2</u>	<u>5</u>
3	2	5	5	1	2	1	4	5	7	1	4	2	1	1
<u>1</u>	3	<u>1</u>	4	7	<u>1</u>	9	<u>1</u>	5	<u>1</u>	<u>2</u>	4	<u>2</u>	<u>6</u>	<u>1</u>

DRILL B—MEDIUM COMBINATIONS

4	3	7	8	6	7	4	9	8	6	2	4	2	6	3
<u>6</u>	3	3	<u>8</u>	<u>2</u>	7	<u>2</u>	9	<u>2</u>	<u>6</u>	<u>6</u>	<u>8</u>	4	5	3
3	2	5	6	9	5	7	2	6	3	8	2	5	7	8
7	5	3	4	9	<u>6</u>	<u>2</u>	<u>8</u>	<u>6</u>	5	4	7	<u>2</u>	7	<u>8</u>

DRILL C—DIFFICULT COMBINATIONS

3	3	7	6	2	5	3	4	5	9	6	5	6	6	4
<u>6</u>	9	<u>8</u>	7	9	9	<u>8</u>	9	7	7	9	<u>8</u>	<u>8</u>	3	7
7	8	8	7	7	9	8	7	9	9	8	9	8	9	9
<u>6</u>	9	5	9	5	<u>2</u>	<u>6</u>	4	<u>8</u>	5	3	4	7	3	<u>6</u>

A seventh-grade class was timed in giving orally the sums in Drill A. The median time for giving the thirty sums was 20 seconds. All pupils who had not finished Drill A in 20 seconds were required to practice on that drill until they had reached the class standard. After finishing Drill A in 20 seconds or less, each pupil was required to reach the same time limit for Drills B and C. The aim of this plan was to cease drill on the easy combinations when a suitable standard had been reached and to concentrate attention on the difficult combinations and have them equally well mastered.

This seventh-grade class was slightly below the Cleveland seventh-grade median for Set A of the Cleveland Arithmetic Test before the drill was given. A year after the drill period was finished (thus allowing time for the immediate effects of the special drill to pass away) this class was again tested on Set A of the Cleveland Arithmetic Test, and the class median was 75 per cent greater than the Cleveland eighth-grade median. This experiment shows the excellent results that can be secured by placing the emphasis where it belongs—upon the difficult combinations and making them equally well mastered with the easy group.

THE RELATIVE DIFFICULTY OF THE MULTIPLICATION FACTS

The relative difficulty of the simple multiplication facts was determined in the same way. Forty-two pupils were tested individually on the 100 simple multiplication facts. Table II shows the median time responses in seconds and also the number of incorrect responses made to each multiplication fact.

A study of Table II shows that there is little relation between the length of the time responses and the number of errors. For example, 9×9 and 8×6 have long time responses and no errors, while 8×0 and 9×0 have short time responses and a large number of errors.

It was observed that several pupils who gave incorrect responses to the first two or three products involving zero as a factor soon saw their error and gave correct responses to the remainder of the products involving zero as a factor. Other pupils who gave incorrect responses to all of the products involving zero as a factor

were shown the principle that "zero times any digit or any digit times zero gives *zero* as a product." In the second presentation of the cards they had no difficulty in giving the correct responses to all the products with zero as a factor. This shows that the

TABLE II*

EASY GROUP			MEDIUM GROUP			DIFFICULT GROUP		
Multiplication Fact	Time Response in Seconds	Number of Errors	Multiplication Fact	Time Response in Seconds	Number of Errors	Multiplication Fact	Time Response in Seconds	Number of Errors
1×7	.96	0	2×2	1.11	0	4×8	1.18	6
6×1	.96	0	0×2	1.11	6	9×5	1.18	3
5×1	.97	1	4×2	1.11	1	3×6	1.18	1
1×6	.97	0	2×7	1.12	0	6×6	1.18	0
1×2	.98	1	4×6	1.12	1	7×6	1.18	2
4×1	.98	1	2×5	1.12	0	3×8	1.19	2
1×4	.99	0	2×4	1.12	2	8×4	1.19	13
9×1	.99	0	0×8	1.13	6	2×8	1.19	1
8×0	.99	7	5×3	1.13	1	8×5	1.19	2
1×3	.99	1	6×4	1.13	1	8×6	1.20	0
9×0	.99	6	5×2	1.13	0	6×7	1.22	2
0×1	.99	6	5×4	1.13	0	5×9	1.24	1
5×0	1.00	6	2×3	1.13	1	8×3	1.27	4
1×8	1.00	1	2×6	1.14	0	6×3	1.27	2
0×4	1.00	6	3×2	1.14	0	7×5	1.28	0
8×1	1.01	1	3×4	1.14	1	9×3	1.30	4
0×5	1.02	9	7×2	1.14	0	3×9	1.30	4
2×0	1.02	6	3×5	1.14	2	8×9	1.30	2
0×7	1.02	9	2×9	1.14	0	7×4	1.30	3
7×1	1.03	0	5×6	1.15	2	4×7	1.30	2
0×6	1.03	5	6×2	1.15	2	4×5	1.30	0
0×3	1.03	6	4×3	1.15	2	4×9	1.31	2
2×1	1.03	1	4×4	1.15	2	6×8	1.32	3
3×1	1.04	0	5×5	1.15	1	8×8	1.32	2
1×1	1.05	1	8×2	1.16	4	9×9	1.32	0
1×0	1.05	7	5×8	1.16	0	7×8	1.32	2
0×9	1.06	7	7×3	1.16	0	8×7	1.34	3
7×0	1.07	6	3×7	1.16	1	9×4	1.37	2
3×0	1.07	8	6×5	1.16	2	9×8	1.37	2
4×0	1.08	5	9×2	1.17	0	7×9	1.40	1
0×0	1.09	0	7×7	1.17	0	9×6	1.50	7
6×0	1.10	7	5×7	1.17	2	6×9	1.53	6
1×5	1.10	2	3×3	1.17	6	9×7	1.55	6
1×9	1.10	1						

*In this table the first digit is used as the multiplier.

products involving zero as a factor are not fundamentally difficult but that the errors are due either to a failure to present the principle or to the lack of sufficient drill to fix the correct responses.

The difficult facts consist mainly of those with 6, 7, 8, or 9 as multipliers. The multiplier 9, occurring seven times in the difficult

group, seems to be the most difficult of all the multipliers. This is possibly due to the 9's being presented last and to the lack of sufficient drill when the pupil was tiring of the work on the multiplication tables. Additional drill must be placed upon these difficult facts to make it possible for the pupils to recall them as quickly as the easy facts. Drills similar to A, B, and C of the addition combinations should be prepared from the table on the multiplication facts and used to make all the multiplication facts equally well mastered.

THE RELATIVE DIFFICULTY OF THE SUBTRACTION FACTS

In conducting a similar experiment to determine the relative difficulty of the simple subtraction facts, twenty-six subjects were tested individually on the 100 simple subtraction facts. Since all of the subjects used in this part of the experiment were adults or older pupils, the median time of response cannot be compared to that of the addition combinations because the younger pupils were included in that part of the experiment and hence increased the length of the time responses. Table III shows the different groups for the subtraction facts.

With three exceptions the difficult group of subtraction facts have the minuends 10 or more than 10. This group corresponds closely to the difficult group in addition. The doubles were found in the easy and medium groups in addition. The corresponding subtraction facts also occur in the easy and medium groups. These facts show the close relation of the addition and subtraction processes.

The majority of the incorrect responses were made to the facts in the difficult group. There does not seem, however, to be any definite relation between the time of response in this group and the number of errors. The total number of errors was approximately 7 per cent of the total number of responses. Over 42 per cent of the responses to 11-9 were incorrect.

When we consider that the subjects were from the seventh- and thirteenth-year classes, this is an extremely high percentage of error and shows that either subtraction is a very difficult process or that we do not spend enough time in teaching the subtraction

facts and in drilling on them. Every teacher should devise drills in the simple subtraction facts so that the difficult facts shall be made equally automatic with the easy ones.

TABLE III

EASY GROUP			MEDIUM GROUP			DIFFICULT GROUP		
Subtraction Fact	Time of Response in Seconds	Number of Errors	Subtraction Fact	Time of Response in Seconds	Number of Errors	Subtraction Fact	Time of Response in Seconds	Number of Errors
4-4	.80	0	5-3	.98	0	10-6	1.11	1
8-8	.81	0	10-8	.98	0	11-4	1.12	3
0-0	.83	0	6-1	.99	1	9-3	1.12	5
5-5	.83	0	8-6	.99	1	8-3	1.12	0
2-2	.84	0	6-5	.99	0	9-6	1.13	2
1-1	.85	0	6-4	.99	0	11-9	1.13	11
7-7	.86	0	8-1	1.00	1	12-5	1.13	5
3-3	.86	0	9-1	1.00	1	12-9	1.14	4
4-2	.86	0	10-9	1.00	0	12-7	1.15	1
6-6	.87	0	9-7	1.00	2	13-9	1.18	5
9-9	.87	1	10-1	1.00	0	13-6	1.20	2
8-7	.88	0	7-3	1.00	2	13-7	1.20	4
9-9	.89	0	11-5	1.00	4	12-3	1.20	5
2-1	.89	1	7-1	1.01	0	11-8	1.20	1
5-4	.90	0	5-2	1.02	1	14-8	1.20	5
8-0	.90	0	10-3	1.02	1	17-8	1.24	4
3-2	.90	1	12-8	1.02	0	14-5	1.25	1
4-3	.91	0	7-5	1.02	1	15-8	1.27	5
3-0	.91	0	9-5	1.03	0	15-6	1.28	4
6-0	.92	1	10-7	1.04	0	11-2	1.28	5
1-0	.92	0	7-2	1.05	2	12-4	1.30	4
10-5	.92	0	16-8	1.05	0	13-5	1.30	4
3-1	.93	0	10-4	1.05	1	11-3	1.30	10
7-0	.93	0	8-2	1.06	2	13-8	1.30	3
5-0	.94	1	6-2	1.06	0	16-9	1.30	9
8-4	.94	0	9-2	1.07	1	15-7	1.35	3
7-6	.94	1	8-5	1.07	0	14-9	1.35	5
4-1	.95	1	9-4	1.08	2	16-7	1.38	8
7-4	.95	0	14-7	1.08	1	15-9	1.38	3
4-0	.96	0	10-2	1.10	1	13-4	1.40	3
2-0	.96	1	18-9	1.10	3	17-9	1.40	4
6-3	.96	0	11-6	1.10	2	14-6	1.50	7
5-1	.97	0	11-7	1.10	6			
12-6	.97	0						
9-8	.97	1						

THE RELATIVE DIFFICULTY OF THE SIMPLE DIVISION FACTS

Eighteen subjects were tested in division facts to determine their relative difficulty. The same apparatus was used for this experiment as that used in the addition combinations. The division facts were printed on the cards in the form: $3\overline{)12}$. Table IV shows the three groups of these facts. This table shows that

the difficult division facts correspond to the difficult multiplication facts, thus showing the close relationship between these two processes.

TABLE IV

EASY GROUP			MEDIUM GROUP			DIFFICULT GROUP		
Division Fact	Time Response in Seconds	Number of Errors	Division Fact	Time Response in Seconds	Number of Errors	Division Fact	Time Response in Seconds	Number of Errors
$4 \div 2$	.88	0	$18 \div 9$	1.01	0	$28 \div 7$	1.11	0
$6 \div 2$	.88	0	$20 \div 4$	1.01	0	$40 \div 8$	1.11	1
$8 \div 2$	.88	0	$30 \div 6$	1.01	1	$54 \div 9$	1.11	0
$10 \div 2$	.90	0	$24 \div 4$	1.01	3	$45 \div 9$	1.13	1
$14 \div 7$	.90	0	$6 \div 2$	1.02	0	$35 \div 5$	1.13	2
$12 \div 3$	.90	0	$9 \div 3$	1.02	4	$40 \div 5$	1.13	1
$12 \div 2$	.90	0	$14 \div 2$	1.03	0	$36 \div 9$	1.13	4
$18 \div 9$	.92	1	$3 \div 1$	1.03	3	$32 \div 8$	1.14	0
$7 \div 1$	.92	1	$20 \div 5$	1.03	0	$64 \div 8$	1.14	1
$4 \div 1$	.93	2	$18 \div 6$	1.03	2	$36 \div 4$	1.15	1
$8 \div 4$	.93	0	$6 \div 3$	1.04	1	$48 \div 8$	1.15	3
$16 \div 8$	.93	0	$15 \div 3$	1.04	0	$81 \div 9$	1.15	1
$8 \div 2$	.94	0	$5 \div 1$	1.05	1	$24 \div 8$	1.15	1
$5 \div 1$	.95	4	$12 \div 4$	1.05	3	$63 \div 7$	1.15	3
$25 \div 5$	.95	0	$49 \div 7$	1.05	1	$45 \div 5$	1.17	1
$10 \div 5$	.95	0	$21 \div 7$	1.05	1	$54 \div 6$	1.17	3
$2 \div 1$	.95	3	$35 \div 7$	1.07	1	$48 \div 6$	1.20	3
$10 \div 2$	.96	0	$21 \div 3$	1.07	2	$72 \div 9$	1.20	3
$9 \div 3$	.98	0	$42 \div 7$	1.08	4	$24 \div 3$	1.20	2
$8 \div 8$	.98	2	$27 \div 9$	1.08	1	$28 \div 4$	1.20	6
$16 \div 4$	.98	0	$18 \div 2$	1.08	1	$72 \div 8$	1.23	2
$8 \div 2$	1.00	0	$36 \div 6$	1.08	2	$32 \div 4$	1.25	2
$6 \div 6$	1.00	0	$16 \div 2$	1.10	0	$56 \div 7$	1.25	4
$12 \div 6$	1.00	0	$18 \div 3$	1.10	2	$27 \div 3$	1.25	3
$12 \div 3$	1.00	0	$24 \div 6$	1.10	2	$63 \div 9$	1.27	5
$15 \div 5$	1.00	1	$30 \div 5$	1.10	0	$56 \div 8$	1.30	4
$6 \div 2$	1.00	0	$42 \div 6$	1.10	3			

Another significant point in the table is the large number of errors for the easy divisions: $4 \div 4$; $5 \div 5$; $8 \div 8$; and $9 \div 9$. The incorrect response of 0 was given in many cases. These errors were probably due to lack of drill on such numbers.

The complete tables for each of the arithmetical processes are included in this article so that teachers of arithmetic may form drills similar to those given in connection with the addition combinations. By means of these groups we can concentrate the drill where it belongs—on the difficult facts—and thus conserve the time of the pupils to the best advantage.